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The supersymmetric soliton correlation matrix

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Abstract. *Supersymmetric systems in* (2/2) *dimensions integrable by the supersymmetric generalization of the Zakharov-Shabat «dressing» method are studied. The supersymmetric version of the «soliton correlation matrix» is used to obtain multi-soliton solutions to generic supersymmetric systems of Zakharov-Mikhailov-Shabat type, together with their reductions under finite automorphism groups. The sypersymmetric* S^2 *sigma model is worked out as an explicit application of the method.*

1. INTRODUCTION

In a recent work [1], multi-soliton solutions to two dimensional integrable systems were studied through a new approach based upon the «soliton correlation matrix». This approach begins with the «dressing method» of Zakharov, Mikhailov and Shabat [2, 3] (henceforth, ZMS), but brings out new geometrical structure relating to the linearization of multi-Bäcklund transformations and

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nonlinear superposition of solitons. Combined with the reduction programme of Mikhailov [4], this leads to explicit multi-soliton solutions to a wide class of problems, including all sigma models with values in Riemannian Symmetric Spaces [5], the non-Abelian two dimensional Toda Lattice [1.4] and the Thirring model [6].

Numerous two-dimensional models also admit natural supersymmetric extensions; e.g. the sine-Gordon equation [7, 8], the two-dimensional Toda Lattice [9] and the sigma models [10, 12]. For the principal $SU(n)$ sigma model, Mikhailov [11] has shown that the supersymmetric case may be solved using only a formal modification of the ZMS techniques, although the sense of «classical» solutions involving anti-commuting fields is still subject to interpretation. It is our purpose here to give the supersymmetric generalization of the methods developed in ref. (1,5), thereby allowing the determination of multi-soliton solutions to all supersymmetric extensions of integrable systems of ZMS type.

In the following section, we define the supersymmetric ZMS systems in their generic form, together with the allowed reductions, and give the supersymmetric form of the dressing method for solitons. Next, we introduce the soliton correlation matrix (or M-matrix) formalism suitably «supersymmetrized», the linearization method and the solution in terms of both the M-matrix and the residues defining the ZMS «dressing matrix». In the final section, the results are illustrated in detail by the example of soliton solutions for the supersymmetric extension of the S^2 sigma model.

Since all the results are obtained through merely formal modifications of the corresponding nonsupersymmetric ones, as developed in refs. (1) and (5), we give no proofs, only indicating where differences arise.

2. SUPERSYMMETRIC ZMS SYSTEMS AND THE DRESSING METHOD

All fields will be defined on a (2/2) dimensional superspace extension of 2--dimensional Minkowski space with co-ordinates $(\xi, \eta, \theta_1, \theta_2)$, where (ξ, η) are the usual light-cone co-ordinates and (θ_1, θ_2) are anti-commuting real spinorial co-ordinates.

For fermionic (odd-graded) components of super fields with matrix values, we have the following conventions for complex-conjugation, transpose and hermitian adjoint, respectively:

$$
\begin{aligned}
&(\overline{AB}) = -\overline{A}\,\overline{B} \\
(2.1)\qquad &(AB)^T = -B^T A^T \\
&(AB)^+ = B^+ A^+.
\end{aligned}
$$

The super fields themselves will always be taken as homogenous (even or odd), hence admitting the component-wise expansions:

(2.2) $$\hat{g} = g + i\,\theta_2 g_1 - i\,\theta_1 g_2 + i\,\theta_1\theta_2 G$$

where (g, G) are matrix-valued component fields of the same parity as the matrix-valued superfield $\hat{g}$, and (g_1, g_2) are of opposite parity. Component fields depend only upon the ordinary Minkowski space co-ordinates (ξ, η).

Invertible even parity superfields may be expressed in the form:

(2.3) $$\hat{g} = g + i\,\theta_2\Lambda_1 g - i\,\theta_1\Lambda_2 g + i\,\theta_1\theta_2 F g$$

where $g(\xi, \eta)$ is invertible, the inverse of $\hat{g}$ being:

(2.4) $$\hat{g}^{-1} = g^{-1} - i\,\theta_2 g^{-1}\Lambda_1 + i\,\theta_1 g^{-1}\Lambda_2 - i\,\theta_1\theta_2 g^{-1}\{F + i[\Lambda_1, \Lambda_2]\}.$$

For even or odd parity superfields $\hat{g}$, the corresponding superfields $\bar{\hat{g}}$, $\hat{g}^+$, $\hat{g}^T$ are defined consistently with (2.1). Whereas (θ_1, θ_2) are taken as real, complex conjugation, according to (2.1) inverts order and hence, $\overline{\theta_1\theta_2} = -\bar{\theta}_1\bar{\theta}_2$. We denote the fermionic left-translation derivations, following Mikhailov [11]:

(2.5) $$D_1 \equiv \frac{\partial}{\partial\theta_2} - i\,\theta_2\frac{\partial}{\partial\xi}$$
$$D_2 \equiv -\frac{\partial}{\partial\theta_1} + i\,\theta_1\frac{\partial}{\partial\eta}.$$

With these conventions, we have:

(2.6) $$D_i(\bar{\hat{g}}) = -(\overline{D_i\hat{g}}), \quad D_i(\hat{g})^T = (D_i\hat{g})^T$$
$$D_i(\hat{g})^+ = -(D_i\hat{g})^+, \quad D_i\hat{g}^{-1} = -\hat{g}^{-1}D_i\hat{g}\,\hat{g}^{-1}$$
$$i = 1, 2.$$

The supersymmetric ZMS system is:

(2.7) $$D_1\hat{\psi} = \hat{U}\hat{\psi}$$
$$D_2\hat{\psi} = \hat{V}\hat{\psi}$$

where $\hat{\psi}$ is an even, invertible matrix superfield depending on a complex parameter λ and $\{\hat{U}, \hat{V}\}$ are odd matrix superfields which are meromorphic in λ with fixed poles on the Riemann sphere. The integrability conditions for (2.7) are:

(2.8) $$D_2\hat{U} + D_1\hat{V} - \{\hat{U}, \hat{V}\} = 0$$

where { , } denotes anti-commutation. These relations must be satisfied identically in λ and represent the supersymmetric field equations under study.

Expanding $\hat{U}$ and $\hat{V}$ in components:

$$\hat{U} = \chi_1 + u_1\theta_1 + u_2\theta_2 + i\,\theta_1\theta_2\chi_2 \tag{2.9}$$
$$\hat{V} = \phi_1 + v_1\theta_1 + v_2\theta_2 + i\,\theta_1\theta_2\phi_2$$

(where each term in $\hat{U}$ and in $\hat{V}$ has the same set of poles in λ), eq. (2.8) is equivalent to the system:

$$\begin{aligned} & u_1 - v_2 + \{\chi_1, \phi_1\} = 0 \\ & i\chi_{1\eta} - i\phi_2 + [\chi_1, v_1] + [\phi_1, u_1] = 0 \\ & -i\phi_{1\xi} - i\chi_2 + [\chi_1, v_2] + [\phi_1, u_2] = 0 \\ & u_{2\eta} + v_{1\xi} + i[u_1, v_2] + i[v_1, u_2] - \{\chi_1, \phi_2\} - \{\phi_1, \chi_2\} = 0 \end{aligned} \tag{2.10}$$

interpreted, again, as holding identically in λ.

As usual, the system (2.7), (2.8) admits gauge transformations of the type:

$$\begin{aligned} & \hat{\psi} \longrightarrow \hat{f}\hat{\psi} \\ & \hat{U} \longrightarrow \hat{f}\hat{U}\hat{f}^{-1} + D_1\hat{f}\hat{f}^{-1} \\ & \hat{V} \longrightarrow \hat{f}\hat{V}\hat{f}^{-1} + D_2\hat{f}\hat{f}^{-1} \end{aligned} \tag{2.11}$$

where $\hat{f}(\xi, \eta, \theta_1, \theta_2)$ is any invertible matrix superfield.

As in the nonsupersymmetric case, the «generic» system is determined by the meromorphic structure of $(\hat{U}, \hat{V})$ and the choice of gauge. To obtain interesting reduced systems, we require the fields to be invariant under an additional discrete or finite group of automorphisms of (2.7) consisting of transofrmations:

$$\begin{aligned} & \text{(a)} \quad \hat{\psi}(\lambda) \longrightarrow \tilde{\hat{\psi}}(\lambda) \equiv \hat{f}\sigma[\hat{\psi}(\widetilde{S^{-1}}(\lambda))] \\ & \text{(b)} \quad \hat{U}(\lambda) \longrightarrow \tilde{\hat{U}}(\lambda) = \pm\hat{f}\sigma_*[\hat{U}(\widetilde{S^{-1}}(\lambda))]\hat{f}^{-1} + D_1\hat{f}\hat{f}^{-1} \\ & \text{(c)} \quad \hat{V}(\lambda) \longrightarrow \tilde{\hat{V}}(\lambda) = \pm\hat{f}\sigma_*[\hat{V}(\widetilde{S^{-1}}(\lambda))]\hat{f}^{-1} + D_2\hat{f}\hat{f}^{-1} \\ & \qquad \sigma \in \operatorname{Aut} GL(n, \mathbb{C}). \end{aligned} \tag{2.12}$$

Here S is a conformal map on $\mathbb{C}P^1$, identified (up to sign) with an element of $SL(2, \mathbb{C})$

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1 \tag{2.13}$$

acting, as usual, by linear fractional transformations

$$S : \lambda \longrightarrow \frac{a\lambda + b}{c\lambda + d} \tag{2.14}$$

and preserving the singularity sets of $\hat{U}$ and $\hat{V}$. The automorphism σ may by any of the following four types:

$$\begin{aligned} \sigma_1(g) &= t g t^{-1} & \sigma_3(g) &= t g^{T-1} t^{-1} \\ \sigma_2(g) &= t \bar{g} t^{-1} & \sigma_4(g) &= t g^{+-1} t^{-1} \end{aligned} \tag{2.15}$$

$$t, g \in GL(n, \mathbb{C})$$

where t is some fixed element of $GL(n, \mathbb{C})$ and σ_* denotes its differential at the identity.

The gauge transformation function $\hat{f}$ in general depends upon (S, σ) and the choice of the final gauge. The tilde in $\widetilde{S^{-1}}(\lambda)$ denotes complex conjugation for cases σ_2 and σ_4 only, so that the entire transformation (2.12) is holomorphic. Similarly, the upper choice of sign $\pm$ in (2.12 b,c) applies for type σ_1, σ_3 and the lower for σ_2, σ_4 to maintain consistency with the anti-hermitian nature of the fermionic derivations D_i in (2.7).

As an illustrative example, we may consider the supersymmetric chiral sigma models and their reductions. The ZMS system (2.7) for this case is [11]:

$$\begin{aligned} D_1 \hat{\psi} &= \frac{\hat{A}}{1+\lambda} \hat{\psi} \\ D_2 \hat{\psi} &= \frac{\hat{B}}{1-\lambda} \hat{\psi} \end{aligned} \tag{2.16}$$

with integrability conditions:

$$\begin{aligned} & D_2 \hat{A} - D_1 \hat{B} = 0 \\ \text{where} \quad & \hat{A} \equiv (D_1 \hat{g}) \hat{g}^{-1}, \qquad \hat{B} \equiv (D_2 \hat{g}) \hat{g}^{-1} \\ & \hat{g} \equiv \hat{\psi}(\lambda = 0). \end{aligned} \tag{2.17}$$

The reduction, for example, to $G_p(\mathbb{C}^{p+n})$ Grassmannian sigma-models is given by the invariance conditions [1]

$$\begin{aligned} & I_{p,n} \hat{\psi}(\lambda) I_{p,n}^{-1} = \hat{g}\, \hat{\psi}(1/\lambda) \\ & \hat{\psi}^+(\bar{\lambda}) = \hat{\psi}^{-1}(\lambda) \end{aligned} \tag{2.18}$$

$$\text{where} \quad I_{p,n} = \operatorname{diag}\{\underbrace{+1 \dots +1}_{p}, \underbrace{-1 \dots -1}_{n}\}$$

which is equivalent to choosing $\hat{g}$ in the form:

$$\hat{g} = I_{p,n}(2\hat{P} - \mathbb{1}). \tag{2.19}$$

where $\hat{P}$ is a rank p orthogonal projector. The supersymmetric reduced field equations become:

$$[D_2 D_1 \hat{P}, \hat{P}] = 0. \tag{2.20}$$

Further reduction by the reality condition:

$$\hat{\psi}(\bar{\lambda}) = \bar{\hat{\psi}}(\lambda) \tag{2.21}$$

implies that $\hat{P}$ is a real projector.

In particular for $p = 1$, representing the rank-1 projector in terms of a unit vector $\hat{n}$ as $\hat{P} = \hat{n}\,\hat{n}^T$, this gives the equations of the S^n supersymmetric sigma model:

$$D_1 D_2 \hat{n} + (D_1 \hat{n}, D_2 \hat{n})\,\hat{n} = 0. \tag{2.22}$$

Expanding in component fields:

$$\hat{n} = n_1 + i\,\theta_2 \chi_1 - i\,\theta_1 \chi_2 + i\,\theta_1 \theta_2 n_2 \tag{2.23}$$

where $(n_1, n_2) = i(\chi_1, \chi_2)$, $\quad (n_1, n_1) = 1$, $\quad (n_1, \chi_1) = (n_1, \chi_2) = 0$

gives the equivalent system:

$$\begin{aligned} &\chi_{1\eta} + (n_{1\eta}, \chi_1)\, n_1 + i(\chi_1, \chi_2)\,\chi_2 = 0 \\ &\chi_{2\xi} + (n_{1\xi}, \chi_2)\, n_1 - i(\chi_1, \chi_2)\,\chi_1 = 0 \\ &n_{1\xi\eta} + (n_{1\xi}, n_{1\eta})\, n_1 + i(n_{1\eta}, \chi_1)\,\chi_1 + i(n_{1\xi}, \chi_2)\,\chi_2 = 0 \end{aligned} \tag{2.24}$$

with n_2 determined as:

$$n_2 = i(\chi_1, \chi_2)\, n_1 . \tag{2.25}$$

The dressing method of Zakharov-Mikhailov-Shabat [2, 3] proceeds exactly as in the nonsupersymmetric case. Namely we introduce an even superfield «dressing matrix» $\hat{\chi}(\lambda)$ which is meromorphic on $\mathbb{C}P^1$ with simple fixed poles at $\{\lambda_i\}_{i=1}^{K}$, invertible except at $\{\lambda_i, \mu_i\}_{i=1}^{K}$, where $\{\mu_i\}$ are the poles of $\hat{\chi}^{-1}(\mu)$, which are also assumed simple and fixed. If we choose a gauge such that $\hat{\chi}(\infty) = \hat{\chi}^{-1}(\infty) = \mathbb{1}$, we way express these as:

$$\text{(a)} \quad \hat{\chi}(\lambda) = \mathbb{1} + \sum_{i=1}^{K} \frac{\hat{Q}_i}{\lambda - \lambda_i} \tag{2.26}$$

(2.26) (b) $$\hat{\chi}^{-1}(\lambda) = \mathbb{1} + \sum_{i=1}^{K} \frac{\hat{R}_i}{\lambda - \mu_i} .$$

Given any solution $(\hat{\psi}_0, \hat{U}_0, \hat{V}_0)$ of the system (2.7), (2.8), a new solution is generated by the multi-Bäcklund transformation:

(2.27) $$\begin{aligned} \hat{\psi}_0 &\longrightarrow \hat{\chi}\,\hat{\psi}_0 \equiv \hat{\psi} \\ \hat{U}_0 &\longrightarrow \hat{\chi}\,\hat{U}_0 \hat{\chi}^{-1} + D_1 \hat{\chi}\,\hat{\chi}^{-1} \equiv \hat{U} \\ \hat{V}_0 &\longrightarrow \hat{\chi}\,\hat{V}_0 \hat{\chi}^{-1} + D_2 \hat{\chi}\,\hat{\chi}^{-1} \equiv \hat{V} \end{aligned}$$

provided the residue superfields $\{\hat{Q}_i, \hat{R}_i\}$ satisfy the fermionic differential equations:

(2.28) $$\begin{aligned} D_1 \hat{Q}_i &= \hat{U}(\lambda_i)\,\hat{Q}_i - \hat{Q}_i\,\hat{U}_0(\lambda_i) \\ D_2 \hat{Q}_i &= \hat{V}(\lambda_i)\,\hat{Q}_i - \hat{Q}_i\,\hat{V}_0(\lambda_i) \\ D_1 \hat{R}_i &= \hat{U}_0(\mu_i)\,\hat{R}_i - \hat{R}_i\,\hat{U}(\mu_i) \\ D_2 \hat{R}_i &= \hat{V}_0(\mu_i)\,\hat{R}_i - \hat{R}_i\,\hat{V}(\mu_i) \end{aligned}$$

which assure the preservation of the meromorphic structure of $\{\hat{U}_0, \hat{V}_0\}$. Moreover, for consistency with (2.26 a, b), the residues $\{\hat{Q}_i, \hat{R}_i\}$ must satisfy the quadratic constraints:

(2.29.a) $$\hat{Q}_i + \hat{R}_i + \sum_{j \neq i}^{K} \left(\frac{\hat{R}_j \hat{Q}_i}{\lambda_i - \mu_j} + \frac{\hat{R}_i \hat{Q}_j}{\mu_i - \lambda_j} \right) = 0$$

$$\hat{R}_i \hat{Q}_i = (\lambda_i - \mu_i) \left[\hat{R}_i + \sum_{j \neq i}^{K} \frac{\hat{R}_i \hat{Q}_j}{\mu_i - \lambda_j} \right] =$$

(2.29.b) $$= -(\lambda_i - \mu_i) \left[\hat{Q}_i + \sum_{j \neq i}^{K} \frac{\hat{R}_j \hat{Q}_i}{\lambda_i - \mu_j} \right].$$

For reduced systems invariant under a group of transformations of the type (2.12) it is necessary that the dressing matrix $\hat{\chi}$ satisfy reduction conditions of the form:

(2.30) $$\sigma[\hat{\chi}(\lambda)] = \tilde{\tilde{f}}^{-1} \hat{\chi}(S(\tilde{\lambda}))\, \hat{f}$$

where $\tilde{\tilde{f}}$ depends on the gauge of the new solution, defining reductions of the same form as (2.12). This implies in particular that the set of poles of $\hat{\chi}$ and $\hat{\chi}^{-1}$ must be invariant under the transformations $\lambda \to S(\tilde{\lambda})$.

The introduction of the soliton correlation matrix, which we give in the following section, is mainly a means of solving the system (2.28) by a suitable linearization procedure, as well as the reduction constraints of type (2.30), exactly as for the non-supersymmetric case.

3. THE SUPERSYMMETRIC SOLITON CORRELATION MATRIX

Proceeding exactly as in refs.(1) and (5), we introduce the soliton correlation matrix as the $nK \times nK$ dimensional matrix $\hat{M}$ whose ij^{th} $(1 \leqslant i, j \leqslant K)$ $n \times n$ block is:

$$\hat{M}_{ij} \equiv \frac{1}{(2\pi i)^2} \oint_{\mu_i} d\mu \oint_{\lambda_j} d\lambda \, \frac{\hat{\chi}^{-1}(\mu)\,\hat{\chi}(\lambda)}{\mu - \lambda} \tag{3.1}$$

where the contour integrals are around paths containing only the poles indicated.

Without loss of generality, the only degeneracies allowed among pole locations will be on the diagonal ($\mu_i = \lambda_i$). If $\mu_i \neq \lambda_j$, (3.1) reduces to

$$\hat{M}_{ij} = \frac{\hat{R}_i \hat{Q}_j}{\mu_i - \lambda_j} \tag{3.2}$$

and if $\mu_i = \lambda_i$, the $\hat{M}_{ii}$ is given by sums over suitable bilinear combinations of $\hat{R}_i$'s and $\hat{Q}_j$'s, such that the constraints may equivalently be expressed:

$$\mu_i \hat{M}_{ij} - \lambda_j \hat{M}_{ij} = -\sum_{kl} \hat{M}_{ik} \hat{M}_{lj}. \tag{3.3}$$

The residues $\hat{Q}_i$, $\hat{R}_i$ are uniquely determined from $\hat{M}$ by summing over row or column blocks:

$$\begin{aligned} \hat{Q}_i &= \sum_{j=1}^{K} \hat{M}_{ji} \\ \hat{R}_i &= -\sum_{j=1}^{K} \hat{M}_{ij}. \end{aligned} \tag{3.4}$$

The differential equations (2.28) are equivalent to the fermionic Riccati system:

$$\begin{aligned} D_1 \hat{M} &= \hat{p}^{+} \hat{M} - \hat{M} \hat{s}^{+} - \hat{M} \hat{r}^{+} \hat{M} \\ D_2 \hat{M} &= \hat{p}^{-} \hat{M} - \hat{M} \hat{s}^{-} - \hat{M} \hat{r}^{-} \hat{M} \end{aligned} \tag{3.5}$$

where $\{\hat{p}^{\pm}, \hat{s}^{\pm}, \hat{r}^{\pm}\}$ are $(nK \times nK)$ fermionic matrices determined in terms of their $n \times n$ blocks as:

$$\hat{p}^+ \equiv \operatorname{diag}\{\hat{U}_0(\mu_i)\} \qquad \hat{p}^- \equiv \operatorname{diag}\{\hat{V}_0(\mu_i)\}$$

$$\hat{s}^+ \equiv \operatorname{diag}\{\hat{U}_0(\lambda_i)\} \qquad \hat{s}^- \equiv \operatorname{diag}\{\hat{V}_0(\lambda_i)\}$$

(3.6)
$$\hat{r}^+_{ij} \equiv -\frac{\hat{U}_0(\mu_j) - \hat{U}_0(\lambda_i)}{\mu_j - \lambda_i} \qquad \hat{r}^-_{ij} \equiv -\frac{\hat{V}_0(\mu_j) - \hat{V}_0(\lambda_i)}{\mu_j - \lambda_i}$$

if $\mu_j \neq \lambda_i$

$$\hat{r}^+_{ii} \equiv -U_0'(\lambda_i = \mu_i) \qquad \hat{r}^-_{ii} \equiv -\hat{V}_0'(\lambda_i = \mu_i)$$

if $\lambda_i = \mu_i$.

The reduction conditions (2.30) are equivalent to contraints of the following type on the $\hat{M}$-matrix:
For automorphisms of form σ_1:

$$\hat{M}_{s(i)s(j)} = \frac{\hat{f}t\,\hat{M}_{ij}\,(\hat{f}t)^{-1}}{(c\mu_i + \mathrm{d})(c\lambda_j + \mathrm{d})}$$

For σ_2:

$$\hat{M}_{s(\bar{i})s(\bar{j})} = \frac{\hat{f}t\,\hat{M}_{ij}\,(\hat{f}t)^{-1}}{(c\bar{\mu}_i + \mathrm{d})(c\bar{\lambda}_j + \mathrm{d})}$$

(3.7)

For σ_3:

$$\hat{M}_{s(j)s(i)} = \frac{-\hat{f}t\,\hat{M}^T_{ij}\,(\hat{f}t)^{-1}}{(c\mu_i + \mathrm{d})(c\lambda_j + \mathrm{d})}$$

For σ_4:

$$\hat{M}_{s(\bar{j})s(\bar{i})} = \frac{-\hat{f}t\,\hat{M}^+_{ij}\,(\hat{f}t)^{-1}}{(c\bar{\mu}_i + \mathrm{d})(c\bar{\lambda}_j + \mathrm{d})}$$

exactly as in the nonsupersymmetric case [1, 5].

The linearization of (3.5), as well as the proof of consistency of the constraints (3.3) and reduction conditions (3.7) with (3.5) are done exactly the same way as in refs (1, 5) and the details will not be repeated here. The key observation lies in the fact that interpreting the values of $\hat{M}$ as affine co-ordinates on the Grassmann manifold $G_{nk}(\mathbb{C}^{2nK})$, the general solution to (3.5) is obtained by solving a corresponding linear problem for a field G with values in the group $GL(2nK, \mathbb{C})$:

(3.8)
$$D_1\,\hat{G} = \begin{pmatrix} \hat{p}^+ & 0 \\ \hat{r}^+ & \hat{s}^+ \end{pmatrix} \hat{G}$$

$$(3.8) \qquad D_2 \hat{G} = \begin{pmatrix} \hat{p}^- & 0 \\ \hat{r}^- & \hat{s}^- \end{pmatrix} \hat{G}$$

where $\hat{G}$ may be taken in block triangular form:

$$(3.9) \qquad \hat{G} = \begin{pmatrix} \hat{P} & 0 \\ \hat{R} & \hat{S} \end{pmatrix}.$$

Then $\hat{M}$ is given in terms of some constant $nK \times nK$ dimensional matrix m satisfying the constraints (3.3) and the reduction conditions (3.7) (e.g., the value of $\hat{M}$ at some point, if $\hat{G}$ is normalized to unity at that point), by the linear fractional transformation:

$$(3.10) \qquad \hat{M} = \hat{P}\, m [\hat{R}\, m + \hat{S}]^{-1}.$$

The interpretation of the constraints (3.3) and (3.7) in more elegant terms relating to the geometry of $G_{nK}(\mathbb{C}^{2nK})$ is possible, and we refer the reader to refs (1) and (5) for the details which are identical for the supersymmetric case.

The actual solution of (3.8) is accomplished by reducing all matrices to their Jordan normal forms, the result being:

$$\hat{P} = \mathrm{diag}\{\hat{\psi}_0(\mu_i)\}, \quad \hat{S} = \mathrm{diag}\{\hat{\psi}_0(\lambda_i)\}$$

$$\hat{R} = D\,\hat{P} - \hat{S}\,D + \mathrm{diag}\{\hat{\phi}_i\}$$

$$\text{where} \quad D_{ij} = \frac{\mathbb{1}}{\lambda_i - \mu_j} \qquad \text{if} \quad \lambda_i \neq \mu_j$$

$$D_{ii} = 0 \qquad \text{if} \quad \lambda_i = \mu_i$$

$$\hat{\phi}_i = -\hat{\psi}_0'(\lambda_i) + \hat{\psi}_0(\lambda_i)\, C_i \qquad \text{if} \quad \lambda_i = \mu_i$$

$$(C_i = \text{constant})$$

$$= 0 \qquad \text{if} \quad \lambda_i \neq \mu_i.$$

Finally, returning to the residues $\{\hat{Q}_i, \hat{R}_i\}$ determining the dressing matrix $\hat{\chi}(\lambda)$, the solution may be equivalently represented as:

$$\hat{Q}_i = \hat{X}_i\, \hat{F}_i^+$$

$$\hat{R}_i = \hat{H}_i\, \hat{K}_i^+$$

where $\hat{X}_i, \hat{F}_i, \hat{H}_i, \hat{K}_i$ are rectangular matrices of maximal rank with $(\hat{F}_i, \hat{H}_i)$ determined in terms of similar constant matrices $\{f_i, h_i\}$ by:

$$\hat{F}_i = \hat{\psi}_0^{+\,-1}(\lambda_i)\, f_i \qquad \hat{H}_i = \hat{\psi}_0(\mu_i)\, h_i$$

and $(\hat{X}_i, \hat{K}_i)$ by solving the linear systems:

$$\sum_i \hat{X}_i \hat{\Gamma}_{ij} = \hat{H}_j \qquad \sum_i \hat{K}_i \hat{\Gamma}^+_{ij} = \hat{F}_j$$

$$\text{where} \quad \hat{\Gamma}_{ij} = \frac{\hat{F}^+_i \hat{H}_j}{\lambda_i - \mu_j} \qquad \text{if} \quad \lambda_i \neq \mu_j$$

$$\hat{\Gamma}_{ii} = \hat{F}^+_i \hat{\phi}_i \hat{H}_i \qquad \text{if} \quad \lambda_i = \mu_i$$

$$\text{and} \quad \hat{F}^+_i \hat{H}_i = 0.$$

The constraints (3.7) determining the reductions may also be expressed directly in terms of $\{\hat{F}_i, \hat{H}_i, \hat{\Gamma}_{ij}\}$, but since the relations are identical to those in ref. (1), we refer the reader there for further details.

4. AN EXAMPLE: SUPER-SYMMETRIC S^2 MODEL

As an illustration of the above results, consider the system (2.22) - (2.24) for the case where $\hat{n}$ takes values on the unit sphere $S^2 \subset \mathbb{R}^3$. In general, the poles in $\hat{\chi}$ and $\hat{\chi}^{-1}$ must occur in sets of four $\{\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i\}$ because of the reduction conditions (2.18), (2.21). However, we may simplify to pairs of poles by choosing them on the unit circle $|\lambda_i| = 1$. Moreover, without loss of generality, we may take the residues to be of rank 1 (since rank 1 and rank 2 give equivalent solutions). It follows (cf. ref. (5)) that the multi-soliton solutions may be obtained from a given «vacuum» solution $(\hat{\psi}_0, \hat{A}_0, \hat{B}_0)$ of eq. (2.16) satisfying the reduction conditions (2.18), (2.21) through the transformation (2.27), with $\hat{\chi}$ of the form:

$$\hat{\chi} = \mathbb{1} + \sum_{i=1}^{\ell} \left[\frac{\hat{X}_i \hat{F}^+_i}{\lambda - \lambda_i} + \frac{\hat{X}_i \hat{F}^T_i}{\lambda - \bar{\lambda}_i} \right] \tag{4.1}$$

$$\text{where} \quad \hat{F}_i = \bar{\hat{\psi}}_0(\lambda_i) f_i, \tag{4.2}$$

the constant 3-vector f_i is such that $\hat{F}_i$ satisfies:

$$\hat{F}^T_i \hat{F}_i = 0, \qquad \bar{\hat{F}}_i = \hat{\psi}_0(\lambda = 0) \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \hat{F}_i. \tag{4.3}$$

(it being sufficient to impose this at some arbitrary initial point) and the vectors $\hat{X}_i$ are obtained by solving the linear system:

$$\sum_{\substack{i=1 \\ \neq j}}^{\ell} \left[\frac{\hat{X}_i \hat{F}^+_i \bar{\hat{F}}_j}{\lambda_i - \lambda_j} + \frac{\bar{\hat{X}}_i \hat{F}^T_i \bar{\hat{F}}_j}{\bar{\lambda}_i - \lambda_j} \right] = \bar{\hat{F}}_j. \tag{4.4}$$

The «vacuum» solution is of the form:

$$\hat{A}_0 = i\,\chi_1(\xi) - \theta_2 u_0(\xi)$$
(4.5)
$$\hat{B}_0 = i\,\chi_2(\eta) + \theta_1 v_0(\eta)$$

where (iu_0, iv_0), (χ_1, χ_2) are mutually commuting, respectively even and odd, real anti-symmetric 3×3 matrices. Choosing them as constants, they may be rotated parallel to a fixed $o(3)$ element, such that:

$$u_0 \equiv \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad v_0 = a\,u_0$$

(4.6)
$$i\,\chi_1 = \epsilon_1 u_0, \qquad\qquad i\,\chi_2 = \epsilon_2 u_0$$

where (ϵ_1, ϵ_2) are a pair of anticommuting parameters and a is a real constant which we choose henceforth to be $a = 1$. The vacuum $\hat{\psi}_0$ is, up to initial normalization, determined as:

(4.7)
$$\hat{\psi}_0(\lambda) = \left\{ \mathbb{1} + i\left(\frac{\theta_2\epsilon_1}{1+\lambda} - \frac{\theta_1\epsilon_2}{1-\lambda}\right)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \theta_1\theta_2\,\frac{\epsilon_1\epsilon_2}{1-\lambda^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} \cos z & 0 & \sin z \\ 0 & 1 & 0 \\ -\sin z & 0 & \cos z \end{pmatrix}$$

(4.8)
$$\text{where} \quad z = \frac{\xi}{1+\lambda} + \frac{\eta}{1-\lambda}\,.$$

This gives:

(4.9)
$$\hat{g}_0 = \hat{\psi}_0(0) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}[2\hat{n}_0\hat{n}_0^T - \mathbb{1}]$$

(4.10)
$$\text{where} \quad \hat{n}_0 = \begin{pmatrix} \cos t/2 \\ 0 \\ \sin t/2 \end{pmatrix} + i/2\,(\theta_2\epsilon_1 - \theta_1\epsilon_2)\begin{pmatrix} -\sin t/2 \\ 0 \\ \cos t/2 \end{pmatrix} - \frac{1}{4}\,\theta_1\theta_2\epsilon_1\epsilon_2\begin{pmatrix} \cos t/2 \\ 0 \\ \sin t/2 \end{pmatrix}$$

(4.11)
$$t \equiv \xi + \eta \qquad\qquad x \equiv \xi - \eta.$$

For the case of a single pair of poles at $\lambda = e^{\pm i\phi}$, we obtain the 1-soliton solution in the form (2.23), with:

$$
(4.12)\qquad
\begin{aligned}
&\text{(a)}\quad n_1 = \frac{1}{ch}\begin{bmatrix} \lambda' c\, ch + \lambda'' s\, ch \\ \lambda'' \\ \lambda' s\, ch - \lambda'' c\, sh \end{bmatrix} \\
&\text{(b)}\quad \chi_1 = \frac{\epsilon_1}{2}\begin{bmatrix} -\lambda' s + \lambda'' c\, th - (1-\lambda') s \dfrac{1}{ch^2} \\ (1-\lambda')\, sh/ch^2 \\ \lambda' c + \lambda'' s\, th + (1-\lambda') c\; 1/ch^2 \end{bmatrix} \\
&\text{(c)}\quad \chi_2 = \frac{\epsilon_2}{2}\begin{bmatrix} -\lambda' s + \lambda'' c\, th + (1+\lambda') s\; 1/ch^2 \\ -(1+\lambda')\, sh/ch^2 \\ \lambda' c + \lambda'' s\, th - (1+\lambda') c\; 1/ch^2 \end{bmatrix}
\end{aligned}
$$

$$
(4.13)\qquad
\begin{aligned}
\text{where}\quad & \lambda' = \cos\phi & \lambda'' = \sin\phi \\
& c = \cos t/2 & s = \sin t/2 \\
& sh = \sinh a\,(x - vt), & ch = \cosh a(x - vt) \\
& th = \tanh a(x - vt) & \\
& a = -\frac{1}{2\lambda''} & v = \lambda'.
\end{aligned}
$$

This may be interpreted as the supersymmetric extension of the 1-soliton solution of refs (13 - 14). Indeed, if we set $\chi_1 = \chi_2 = 0$, (4.12) (a) gives precisely the soliton solution of the ordinary (non-supersymmetric) $S^2\,\sigma$-model. Moreover, if we apply the supersymmetry transformation:

$$
(4.14)\qquad n_1 \longrightarrow \exp\,[\epsilon_1 \widetilde{D}_1 - \epsilon_2 \widetilde{D}_2]\, n_1
$$

$$
(4.15)\qquad
\begin{aligned}
\text{where}\quad \widetilde{D}_1 &\equiv \frac{\partial}{\partial\theta_2} + i\theta_2 \frac{\partial}{\partial\xi} \\
\widetilde{D}_2 &\equiv \frac{\partial}{\partial\theta_1} + i\theta_1 \frac{\partial}{\partial\eta}
\end{aligned}
$$

are the supersymmetry generators (right-translations) which anticommute with $\{D_1, D_2\}$, we re-obtain the solution (4.12). More generally, if we expand:

$$(4.16)\qquad \begin{aligned} n_1 &= n + n'\epsilon_1\epsilon_2 \\ \chi_i &= \epsilon_1\chi_i^1 + \epsilon_2\chi_i^2 \qquad i = 1, 2 \end{aligned}$$

in terms of a Grassmann algebra with two generators, we obtain the decoupled σ-model equation for n, together with a set of equations linear in $(n', \chi_i^1\, \chi_i^2)$.

The particular case $n' = 0$, gives again the solution (4.12).

CONCLUSIONS

We have seen that all the results developed in refs. (1, 5) have an immediate extension to the supersymmetric version of Z.M.S. systems, which permits us, with very little modification, to formally obtain the multi-soliton solutions. Unfortunately, without further generalizations, these models tend to decouple into a purely bosonic nonlinear system, together with a set of linear equations determining the fermionic and higher order constituents. Indeed, we have seen that for the detailed example of the supersymmetric $S^2\,\sigma$-model, the 1-soliton solution could be generated from its bosonic part by a supersymmetry transformation. For the supersymmetric Toda Lattice (with values in an ordinary Lie algebra), the decoupling is complete, with the fermions satisfying free field equations. This is not the case however for the more general case where the interactions are defined by the root lattice of a superalgebra [9]. Similar generalizations of σ-models also exist, with the fields taking values in super-symmetric spaces [15]. It would be of considerable interest to determine how general this phenomenon of decoupling and linearization is, and to also extend the present methods to the case where the fields take values in superspaces. Further interesting generalizations would involve supermanifolds with extended supersymmetries as well as higher bosonic dimensions. We leave such considerations to future work.

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